notation

$Q = (Q_0, Q_1)$: quiver $\qquad Q_1 \longrightarrow Q_0 \times Q_0$

Q_0: vertexes, Q_1: edges $\qquad h \mapsto o(h) \to i(h)$

V, W : Q_0-graded finite dimensional complex vector spaces

$$\mathbb{G} := \prod_{i \in Q_0} GL(V_i) \quad , \quad \mathbb{N} := \bigoplus_{h \in Q_1} \mathrm{Hom}(V_{o(h)}, V_{i(h)}) \oplus \bigoplus_{i \in Q_0} \mathrm{Hom}(W_i, V_i)$$

(use \mathbf to avoid conflicts with groups for (another) affine Grassmann)

$\Longrightarrow \mathcal{M}_C = \mathcal{M}_C(\mathbb{G}, \mathbb{N})$ Coulomb branch

$\mathbb{T}_F = \prod_i T(W_i) \times (\mathbb{C}^\times)^{\#\text{loops}}$ flavor symmetry

- $\chi \in \mathrm{Hom}(\mathbb{C}^\times, \underbrace{\pi_1(\mathbb{G})^\wedge}_{\cong\, (\mathbb{C}^\times)^{Q_0} =:\, T^{Q_0}}) \cong \mathrm{Hom}(\mathbb{G}, \mathbb{C}^\times)$ $\qquad \mathbb{C}^\times$-action on $\mathcal{M}_C$

- $\rho \in \mathrm{Hom}(T_F^\vee, \mathbb{C}^\times) \cong \mathrm{Hom}(\mathbb{C}^\times, T_F)$ $\qquad$ GIT parameter for $\mathcal{M}_C(\tilde{\mathbb{G}}, \mathbb{N}) /\!/\!/_\rho\, T_F^\vee$

§3. generalised slices

Assume Q (possibly with (d_i)) is of finite type
$\mathfrak{g}$ = corresponding cpx simple Lie algebra ← as in §2.
G = adjoint type group

$$\lambda := \sum \dim W_i \cdot \varpi_i \quad , \quad \mu = \lambda - \sum \dim V_i \cdot \alpha_i$$

↑ fundamental coweight ↑ simple coroot

<u>Th</u>. $\mathcal{M}_C(\lambda, \mu)$ = generalised affine Grassmannian slice $\overline{\mathcal{W}}^{\lambda}_{\mu}$

I will <u>not</u> give the definition of $\overline{\mathcal{W}}^{\lambda}_{\mu}$.
I will only mention their surprising properties.

Roughly, $\overline{\mathcal{W}}^{\lambda}_{\mu}$ is a moduli space of based maps $\mathbb{P}^1 \dashrightarrow$ flag var.
with singularity at the origin.
type of the singularity is specified by λ.
(Recall $W = 0 \Rightarrow \mathcal{M}_C \cong$ moduli of based maps $\mathbb{P}^1 \longrightarrow$ flag var.)

<u>Rem</u>. • T^{Q_0} is naturally identified with T : maximal torus of G
• deformation by $\Pi_F \longleftrightarrow$ Beilinson – Drinfeld Grassmannian

(1) (Finkelberg–Mirković, Braverman–Finkelberg)

$$\overline{\mathrm{Gr}_G^\lambda} \xleftarrow{p} \overline{W}_\mu^\lambda \xrightarrow{q} Z^{-w_0(\lambda-\mu)}$$

zastava space = partial compactification of moduli of based maps $\mathbb{P}^1 \to G/B$

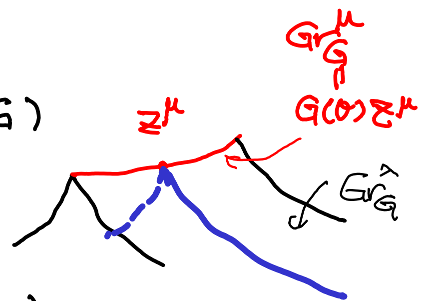
(2) [FM, BF]

Suppose μ: dominant $\Rightarrow$ p is locally closed embedding:

$$\overline{W}_\mu^\lambda \cong \overline{\mathrm{Gr}_G^\lambda} \cap W_{G,\mu} \quad \text{where } W_{G,\mu} = G[z^{-1}]_1 \cdot z^\mu$$

affine Grassmannian slice

$$G[z^{-1}]_1 = \mathrm{Ker}(G[z^{-1}] \xrightarrow{ev_\infty} G)$$



One show that q is birational (even if μ not nec. dominant)

$\Rightarrow$ We have an integrable system and a birational coordinate system induced from $Z^{-w_0(\lambda-\mu)}$. $\Longrightarrow$

$$\begin{tikzcd} \mathcal{M}_C(\lambda,\mu) \arrow[r, dotted, "\Xi"] \arrow[dr] & \overline{W}_\mu^\lambda \arrow[d] \\ & \mathfrak{t}/W \end{tikzcd}$$

▷ Check $\overline{W}_\mu^\lambda$ is correct along $\mathfrak{t}^\bullet/W$

Example type A_1

Suppose $w > v$.

Then

$$\mathcal{M}(v,w) \cong \mathcal{N}_{\mathfrak{sl}(w)} \cap \left\{ \begin{array}{c|c|c} & v & w-v \\ \hline v & \begin{matrix} 1 & & * \\ & \ddots & \vdots \\ & 1 & * \end{matrix} & \begin{matrix} 0 & * \\ & \vdots \end{matrix} \\ \hline w-v & \begin{matrix} * & \cdots & * \\ & 0 & \end{matrix} & \begin{matrix} 1 & & * \\ & \ddots & \vdots \\ & 1 & * \end{matrix} \end{array} \right\}$$

If $v \leq w-v$, it is a variant of Slodowy slice to $\mathbb{O}(v, w-v)$. Mirković–Vybornov.

But it has larger dimension if $v > w-v$.

Rem. When μ : dominant,

$$\overline{\mathcal{W}}^{\lambda}_{\mu} \cong \overline{\mathbb{O}(\lambda)} \cap S(\mu)$$

Mirković–Vybornov

λ, μ are suitably interpreted as partitions quantization (Appendix to the 2nd paper, written by BFN + Kamnitzer, Kodera, Webster, Weekes)

shifted Yangian Y_μ

$E_i^{(q)}, F_i^{(q)}, H_i^{(p)}$ $q>0$ $p \in \mathbb{Z}$

usual relations of Yangian $+ \begin{cases} H_i^{(p)} = 0 & p < -\langle \mu, \alpha_i \rangle \\ H_i^{(-\langle \mu, \alpha_i \rangle)} = 1 \end{cases}$

Th (Weekes 1903.07734)

$$\begin{array}{ccc} Y_\mu & \twoheadrightarrow & \text{quantized Coulomb branch}\,|_{\hbar=1} \\ \downarrow & & \wr \\ & \text{localized ring of difference operators} & \end{array}$$

($\hbar$: variable version holds if Q: finite type

Remembering relations with $H_*^{\mathcal{G}}\left(\left(\frac{\mathcal{G} \times \mathcal{G}}{N(\mathcal{K})}\right)^{\mathcal{T}^\times_{\mathrm{isop}}}\right) \cong$ quiver Hecke algebra $\curvearrowleft$ canonical base

we have

$$\boxed{\text{representation theory of shifted Yangian} \longleftrightarrow \text{canonical bases}}$$

Remark.

Even in non-symmetric cases, quiver Hecke algebra are of symmetric type.

§4. Geometric Satake

Recall

- $\mathcal{M}_C(\lambda,\mu) \cong \overline{W}_\mu^\lambda$
- $\overline{W}_\mu^\lambda \curvearrowleft T^{Q_0} = (\mathbb{C}^\times)^{Q_0} = \pi_1(G)^\wedge$
- $\overline{Gr}_G^\lambda \xleftarrow{p} \overline{W}_\mu^\lambda \xrightarrow{q} Z^{-w_0(\lambda-\mu)}$

$\neq \phi$

Th (Krylov)

(1) $(\overline{W}_\mu^\lambda)^{T^{Q_0}}$ is either single point $\{z^\mu\}$ or ϕ.

irreducible fin. dim. rep. of $G^\vee$ of highest wt $=\lambda$

Furthermore, it is $\neq \phi \iff \mu$ is a weight of $V^{G^\vee}(\lambda)$

(2) Choose $\chi : \mathbb{C}^\times \to T^{Q_0}$ regular dominant cocharacter

repelling set : $(\overline{W}_\mu^\lambda)^{\chi\leq 0} \overset{\text{def.}}{=} \{ x \in \overline{W}_\mu^\lambda \mid \lim_{t\to\infty} \chi(t)x \text{ exists} \} \subset \overline{W}_\mu^\lambda$ closed subvariety

$\Longrightarrow (\overline{W}_\mu^\lambda)^{\chi\leq 0} \cong$ Mirkovic–Vilonen cycle $\overline{Gr}_G^\lambda \cap T_\mu$

use U_-

Remark If we restrict to smooth locus, $(W_\mu^\lambda)^{\chi\leq 0} \subset W_\mu^\lambda$ lagrangian.

Consequences [Mirkovic – Vilonen]

$$H_{top}((\overline{\mathcal{W}}^\lambda_\mu)^{x \leqq 0}) \cong V^{G^\vee}(\lambda)_\mu$$

$$\mathrel{\text{\rotatebox{90}{$\cong$}}}$$

$$\Phi_x(IC(\overline{\mathcal{W}}^\lambda_\mu))$$

hyperbolic restriction functor w.r.t. x

$\Phi_x : \mathrm{Perv}\, \overline{\mathcal{W}}^\lambda_\mu \longrightarrow \mathrm{Vect}$ (in general $D^b(\mathrm{Vect})$, but nonzero degrees vanish)

"hyperbolic semi-smallness"

Many (all?) statements of geometric Satake can be stated in terms of $\overline{\mathcal{W}}^\lambda_\mu$.

Then we could generalise them by using Coulomb branches of quiver gauge theories.

geometric Satake for symmetrizable Kac-Moody Lie algebras

Q : a general quiver without edge loops (with symmetrizers)

$\mathfrak{g}$ = corresponding Kac-Moody algebra , $\mathfrak{g}^\vee$ = Langlands dual

V, W : Q_0 graded v.sp. $\rightsquigarrow$ $\mathbb{G}, \mathbb{N}$: as before

$\mathcal{M}_C(\lambda,\mu)$: Coulomb branch $\qquad\qquad \lambda, \mu$: as before

<u>Conjecture</u>

(1) $\mathcal{M}_C(\lambda,\mu)^{\rho(\mathbb{C}^\times)} = \mathcal{M}_C(\lambda,\mu)^{T^{Q_0}}$ is a single point or ϕ

(2) $\Phi_\chi(IC(\mathcal{M}_C(\lambda,\mu)))$ is concentrated in a single degree, and

Main $\rightarrow$ (3) $\bigoplus_\mu \Phi_\chi(IC(\mathcal{M}_C(\lambda,\mu)))$ has a structure of the integrable highest weight module $V(\lambda) = \bigoplus V(\lambda)_\mu$ of $\mathfrak{g}^\vee$

s.t. Levi restriction is given by hyperbolic restriction w.r.t. singular $\chi_S^\perp$ corresponding to $S \subset Q_0$ subset

Namely

$$\mathfrak{g}^\vee \supset \mathfrak{l} = \mathfrak{l}_S = \mathfrak{t} + \langle e_i, f_i \mid i \in S \rangle$$

$$\Longleftrightarrow \chi_S^\perp : \mathbb{C}^\times \longrightarrow T^{Q_0}$$

- $(\chi_S^\perp)_i = 1$ for $i \in S$
- dominant
- not contained in other hyperplanes

We expect $\mathcal{M}_C(\lambda,\mu)^{\chi_S^\perp} \cong \bigcup_{\lambda'} \mathcal{M}_C(\lambda',\mu)$

Coulomb branch for $Q|_S$

$$\Phi_\chi(IC(\mathcal{M}_C)) \cong \underbrace{\Phi_{\chi_S}}_{\text{representation of } \mathfrak{l}_S} \circ \underbrace{\Phi_{\chi_S^\perp}}_{\text{restriction to the Levi subalgebra}}(IC(\mathcal{M}_C))$$

In particular, we take $S=\{i\}$ and define e_i, f_i by geometric Satake for $\mathfrak{sl}_2$. (MV cycle is $\mathbb{C}^*$ by Ex.)

(4) tensor product is realized by partial resolution $\mathcal{M}_C^\rho$ given by $\rho : \mathbb{C}^\times \longrightarrow T(W) \subset \prod_i GL(W_i)$ (flavor symmetry)

$$W = W^1 \oplus W^2$$

$$t \longmapsto t\,\mathrm{id}_{W^1} \oplus \mathrm{id}_{W^2}$$

Th. Conjecture is true in affine type A

Use bow varieties $\cong \mathcal{M}_C(\lambda,\mu)$

<u>Symplectic duality</u> between quiver varieties vs Coulomb br. for quiver gauge theories

☆ $\chi \in \mathrm{Hom}(\mathbb{C}^\times, \pi_1(G)^\wedge) \cong \mathrm{Hom}(G, \mathbb{C}^\times)$

$$\mathcal{M}_C(\lambda,\mu) \curvearrowleft \chi(\mathbb{C}^\times)$$

$$\begin{array}{c} \mathbf{M}/\!/\!/_{\chi} G \\ \parallel \\ \mathcal{M}_H^{\chi}(\lambda,\mu) \xrightarrow{\pi} \mathcal{M}_H(\lambda,\mu) \end{array}$$

resolution

$$V(\lambda)_\mu \cong \Phi_\chi(IC(\mathcal{M}_C(\lambda,\mu)))$$

[N '98] $H_{top}(\pi^{-1}(0)) \cong V(\lambda)_\mu$

MV cycles

irreducible components of $\pi^{-1}(0)$

☆ $\rho \in \mathrm{Hom}(T_F^\vee, \mathbb{C}^\times) \cong \mathrm{Hom}(\mathbb{C}^\times, T_F)$

$$\mathcal{M}_C^\rho(\lambda,\mu) \longrightarrow \mathcal{M}_C(\lambda,\mu)$$

pushforward

$$\mathcal{M}_H(\lambda,\mu) \curvearrowleft \rho(\mathbb{C}^\times)$$

hyperbolic restriction

$i_\mu : \{\text{fixed pt}\} \hookrightarrow \mathcal{M}_C(\lambda,\mu)$

costalk $H^*(i_\mu^! IC(\mathcal{M}_C(\lambda,\mu)))$ is a graded vector space.

$\rightsquigarrow$ q-analog of weight multiplicity

In fact, it is more natural to compare

$$H^*_{T_{Q_0}}(i_\mu^! IC(\mathcal{M}_C(\lambda,\mu))) \text{ and } H^*_{T_{Q_0}}(\Phi_x(IC(\mathcal{M}_C(\lambda,\mu))) \cong V(\lambda)_\mu \otimes \underbrace{H^*_{T_{Q_0}}(pt)}_{\substack{\| \\ \mathbb{C}[t^{Q_0}] \\ \| \\ \mathbb{C}[(t^\vee)^*]}}$$

<u>Conjecture</u> [Muthiah-N, work in progress]

$$H^*_{T_{Q_0}}(i_\mu^! IC(\mathcal{M}_C(\lambda,\mu))) \cong \left(V(\lambda) \otimes \mathbb{C}[(\mathfrak{g}^\vee/\mathfrak{n}^\vee)^*] \otimes \mathbb{C}_{-\mu}\right)^{B^\vee}$$

This conjecture implies ([by Slofstra])

Assume $\mathfrak{g}$: affine

$$F^i V(\lambda)_\mu \overset{\text{def.}}{=} \{ v \in V(\lambda)_\mu \mid x^{i+1} v = 0 \quad \forall x \in \text{principal Heisenberg subalgebra} \cap \mathfrak{n} \}$$

$$\Longrightarrow H^*(i_\mu^! IC(\mathcal{M}(\lambda,\mu))) \cong \operatorname{gr}^F V(\lambda)_\mu$$ and gr. dim is given by Kostant partition func.

[Braverman-Finkelberg] conjectured a similar statement for $x \rightsquigarrow e$: principal nilpotent, (μ : dominant) but corrected by Slofstra

Th. [Mutliah-N, in progress]
Conj. is true for affine type A.